EXAM GROUP THEORY,
October 28th, 2019, 8:30am–11:30am,
Aletta Jacobshal 01.

Put your name on every sheet of paper you hand in. Please provide complete arguments for each of your answers. The exam consists of 5 questions. You can score up to 7 points for each question, and you obtain 5 points for free.
In this way you will score in total between 5 and 40 points.

(1) Consider $\sigma = (1\ 2\ 3\ 4)(2\ 3\ 4\ 5)(3\ 4\ 5\ 6)] \in S_6$.
   (a) [2 points.] Is σ an even permutation?
   (b) [2 points.] Find the order of σ.
   (c) [3 points.] Compute $\sigma^{28102019}$. - odd number

(2) Let n be a nonzero integer and suppose p is a prime number with the property $p|(2n)^4+1$.
   (a) [2 points.] Show that $2n \bmod p$ is in $(\mathbb{Z}/p\mathbb{Z})^\times$, and that it has order 8 in this group.
   (b) [2 points.] Show that $p \equiv 1 \bmod 8$.
   (c) [3 points.] Show that there exist infinitely many prime numbers $\equiv 1 \bmod 8$.

keer toch...

(3) Let $n \in \mathbb{Z}_{>0}$. In $S_{\mathbb{Z}/n\mathbb{Z}}$, the group consisting of all permutations of the set $\mathbb{Z}/n\mathbb{Z}$, we consider the subgroup G given by

$$G := \{f_{a,b}\colon x \mapsto ax + b \mid a \in (\mathbb{Z}/n\mathbb{Z})^\times ,\ b \in \mathbb{Z}/n\mathbb{Z}\}.$$

In G we have the following two subgroups: $H = \{f_{a,0} \in G \mid a \in (\mathbb{Z}/n\mathbb{Z})^\times\}$, and $N = \{f_{1,b} \in G \mid b \in \mathbb{Z}/n\mathbb{Z}\}$.
   (a) [3 points.] Explain why N is a subgroup of G, and why this subgroup is a normal subgroup.
   (b) [2 points.] Show that HN (defined as the set of all products $f_{a,0} \circ f_{1,b}$) equals G.
   (c) [2 points.] Show that $G/N \cong (\mathbb{Z}/n\mathbb{Z})^\times$.

$n \neq$ prime

(4) This exercise discusses subgroups $H \subset \mathbb{Z}^2$. Let $a, b, c, d \in \mathbb{Z}$.
   (a) [3 points.] Take $H := \mathbb{Z} \cdot (a, b) + \mathbb{Z} \cdot (c, d)$. Show:

$$\mathbb{Z}^2/H \text{ can be generated by a single element } \Leftrightarrow \gcd(a, b, c, d) = 1.$$

   (b) [2 points.] Take $H := \mathbb{Z} \cdot (2, 2) + \mathbb{Z} \cdot (4, 12)$. Compute the order of $(1, 0) + H$ in $\mathbb{Z}^2/H$.
   $= \det(A)$ was shit.
   (c) [2 points.] Again, take $H := \mathbb{Z} \cdot (2, 2) + \mathbb{Z} \cdot (4, 12)$. Calculate the rank and the elementary divisors of $\mathbb{Z}^2/H$.

(5) Let G be a finite group, with $\#G = 2m$ for some odd integer m. Suppose $g \in G$ is an element with $\mathrm{ord}(g) = 2$.
   (a) [2 points.] Explain why indeed such an element g exists in G.
   (b) [2 points.] With S_G the group of all permutations of the set G, let $\lambda_g \in S_G$ be the permutation given by $\lambda_g(h) = gh$ (for any $h \in G$). Show that the sign of the permutation λ_g equals -1.
   (c) [3 points.] Prove that G contains a subgroup of index 2.

$[G:H] = \frac{\#G}{\#H} = 2$

$G \cong_{\text{subgr. of}} S_G$ $h \mapsto gh$